Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(00)00015-0
J. Appl. Maths Mechs, Vol. 63, No. 6, pp. 977–984, 1999

0021-8928/99/\$—see front matter

THE PROPAGATION OF ELASTIC WAVES IN COMPOSITE MATERIALS REINFORCED WITH PIEZOELECTRIC FIBRES†

V. M. LEVIN

Petrozavodsk (e-mail: levin@levin.pgu.karelia.su)

(*Received* 12 *March* 1998)

The propagation of longitudinal elastic waves in composite materials, consisting of a polymer matrix reinforced by continuous fibres in one direction, is considered. The reinforcing fibres have piezoelectric properties and have a thin current-conducting coating ("shunted fibres"). The scattering of electric energy in such materials leads to dispersion of the velocity of the elastic waves and to their attenuation. The effective-field method is used to determine the macroscopic electroelastic constants of such composites. These constants enable one to obtain, in explicit form, the frequency dependence of the real and imaginary parts of the wave number of a longitudinal wave, propagating along the reinforcement direction, and also their dependence on the physical and geometrical characteristics of the components.

To damp mechanical vibrations in structures a method has recently become widely used [1–3] in which piezoelectric elements are employed, which are able to convert mechanical energy into electrical energy and vice versa. One such possibility is based on the use of fibres of PZT piezoelectric ceramics with a thin electrically conducting coating ("shunted fibres") as the reinforcing elements in a composite material. This coating forms a passive electrical circuit, which dissipates the electrical energy, and this damps the elastic vibrations in these materials. A simple one-dimensional model (the "mixture rule") was proposed in [2] for a quantitative description of this effect. Below we use a refined scheme (the effective-field method [4, 5]), which enables the effect of the volume nature of the stress–strain state in the material and the electroelastic coupling on the expression for the effective electroelastic constants to be taken into account.

1. INTEGRAL REPRESENTATIONS OF THE ELECTROELASTIC FIELDS IN COMPOSITES REINFORCED WITH FIBRES AND THE AVERAGING PROCEDURE

Consider a uniform piezoelectric material, maintained under isothermal conditions. The linear constitutive relations for such a material have the form

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{ijk}E_k, \quad D_i = e^T_{ikl}\varepsilon_{kl} + \varepsilon_{ik}E_k \tag{1.1}$$

Here σ and ε are the stress and strain tensors, E and D are the electric field and induction vectors respectively, $C = C^E$ is the elastic moduli tensor for a fixed vector E, $\varepsilon = \varepsilon^\varepsilon$ is the permittivity tensor, $\mathbf{e}$ is the piezoelectric constant tensor, characterizing the coupled electroelastic effects, and the superscript T denotes the operation of transposition.

Relations (1.1) can be conveniently written in the following short form

$$J = \mathbf{L}F, \quad J = \begin{Vmatrix}\sigma \\ D\end{Vmatrix}, \quad \mathbf{L} = \begin{Vmatrix} C & -e \\ e^T & \varepsilon \end{Vmatrix}, \quad F = \begin{Vmatrix}\varepsilon \\ E\end{Vmatrix} \tag{1.2}$$

where the "matrix" $\mathbf{L}$ must be regarded as a linear operator, which converts the tensor-vector pair $[\sigma, D]$ into the analogous pair $[\varepsilon, E]$ and which has symmetry of the electroelastic constants.

The inverse relations to (1.1) can be written in the form

†*Prikl. Mat. Mekh.* Vol. 63, No. 6, pp. 1044–1051, 1999.

$$F = \mathbf{M}J, \quad \mathbf{M} = \begin{Vmatrix} S & d \\ -d^T & \eta \end{Vmatrix} \tag{1.3}$$

$$S = S^D = (C + e\varepsilon^{-1}e^T)^{-1}, \quad \eta = \eta^\sigma = (\varepsilon + e^T C^{-1} e)^{-1}, \quad d = Se\varepsilon^{-1} = C^{-1}e\eta$$

We will now consider a composite material consisting of a uniform matrix with an electroelastic characteristics operator $\mathbf{L}^0$, reinforced with a parallel system of continuous circular cylindrical fibres with electroelastic characteristics $\mathbf{L} = \mathbf{L}^0 + \mathbf{L}^1$. The operator $\mathbf{L}$ can be variable inside each fibre. We will denote the characteristic function of the region V, occupied by the fibres, by $\mathbf{V}(x)$, and the strain tensor by $\boldsymbol{\epsilon}(x)$, and we will represent the electric field vector $E(x)$ as follows [5]:

$$F(x) = F^0(x) + \int \mathbf{P}(x - x')\mathbf{Q}(x')dx' \tag{1.4}$$

$$\mathbf{Q}(x) = \mathbf{L}^1(x)F(x)V(x), \quad \mathbf{L}^1(x) = \mathbf{L}(x) - \mathbf{L}^0$$

Here $F^0 = [\varepsilon^0, E^0]$ are the external elastic and electric fields applied to the medium, which we will henceforth assume to be uniform. The kernel of the integral operator in (1.4) will be expressed in terms of the second derivatives of Green's function $\mathbf{G}(x)$ for the common theory of electroelasticity for a medium with properties $\mathbf{L}^0$ [5]

$$\mathbf{P}(x) = \nabla\mathbf{G}(x)\nabla, \quad \mathbf{G}(x) = \frac{1}{(2\pi)^3}\int \mathbf{G}(k)e^{ik\cdot x}dx \tag{1.5}$$

$$\mathbf{G}(k) = \begin{Vmatrix} G_{ik}(k) & -\gamma_i(k) \\ \gamma_k(k) & g(k) \end{Vmatrix}$$

$$G_{ij} = \left(\Lambda_{ij} + \frac{1}{\lambda}h_i h_j\right)^{-1}, \quad \gamma_j = \frac{1}{\lambda}h_l G_{lj}, \quad g = -(\lambda + h_i \Lambda_{ij}^{-1} h_j)^{-1}$$

$$\Lambda_{il}(k) = C^0_{ijkl}k_j k_l, \quad h_i(k) = e^0_{ikl}k_k k_l, \quad \lambda(k) = \varepsilon^0_{ik}k_i k_k$$

Like (1.4), the stress field and the electric displacement field $J = [\sigma, D]$ can be represented in the same way as (1.4)

$$J(x) = J^0(x) + \int \mathbf{R}(x - x')\mathbf{M}^0\mathbf{Q}(x')dx' \tag{1.6}$$

$$J^0(x) = \mathbf{L}^0 F^0(x), \quad \mathbf{R}(x) = -\mathbf{L}^0\delta(x) - \mathbf{L}^0\mathbf{P}(x)\mathbf{L}^0$$

where $\delta(x)$ is the Direc delta function.

We will choose the system of coordinates in such a way that the x_3 axis coincides with the direction of the reinforcement. Since the perturbation of the electroelastic characteristics $\mathbf{L}^1$ and also the elastic and electric fields inside the fibres (in the case of constant external fields F^0) depend only on the x_1 and x_2 coordinates, integral representations (1.4) and (1.6) take the form

$$F(\bar{x}) = F^0 + \int \hat{\mathbf{P}}(\bar{x} - \bar{x}')\mathbf{Q}(\bar{x}')d\bar{x}', \quad \bar{x} = \bar{x}(x_1, x_2)$$

$$J(\bar{x}) = J^0 + \int \hat{\mathbf{R}}(\bar{x} - \bar{x}')\mathbf{M}^0\mathbf{Q}(\bar{x}')d\bar{x}' \tag{1.7}$$

The kernel $\hat{\mathbf{P}}(\bar{k})$ is related to the function $\mathbf{P}(x)$ in (1.5) by the equation

$$\hat{\mathbf{P}}(x_1, x_2) = \int_{-\infty}^{\infty} \mathbf{P}(x_1, x_2, x_3)dx_3 \tag{1.8}$$

Hence it follows that the symbol of the operator $\hat{\mathbf{P}}$ (the Fourier transformation $\hat{\mathbf{P}}(\bar{k})$ of the function $\hat{\mathbf{P}}(k)$) is given by the expression

$$\hat{\mathbf{P}}(\bar{k}) = k\mathbf{G}(k)k\,|_{k_3=0}, \quad \bar{k} = \bar{k}(k_1, k_2) \tag{1.9}$$

where the function $\mathbf{G}(k)$ is defined in (1.5).

Suppose now that the set of fibres is random and uniformly distributed in space. Then the problem of averaging consists of determining the electroelastic fields at an arbitrary point x of the composite material, averaged over the ensemble of samples of this set. Starting from relations (1.7), we can write the expressions for these averages in the form

$$\langle F(\bar{x})\rangle = F^0 + \int \hat{\mathbf{P}}(\bar{x}-\bar{x}')\langle \mathbf{Q}(\bar{x}')\rangle d\bar{x}' \qquad (1.10)$$
$$\langle J(\bar{x})\rangle = J^0 + \int \hat{\mathbf{R}}(\bar{x}-\bar{x}')\mathbf{M}^0\langle \mathbf{Q}(\bar{x}')\rangle d\bar{x}'$$

Here we have taken into account the fact that $\hat{\mathbf{P}}(\bar{x})$ and $\hat{\mathbf{R}}(\bar{x})$ are deterministic functions. If the set of fibres is spatially uniform, $F(\bar{x})$, $J(\bar{x})$ and $Q(\bar{x})$ are uniform random functions, which possess the property of ergodicity. Therefore, for example, the mean $\langle \mathbf{Q}(x)\rangle$ is a constant quantity, the value of which can be found from a typical fixed sample of the function $\mathbf{Q}(\bar{x})$

$$\langle \mathbf{Q}\rangle = \lim_{S\to\infty} \frac{1}{S}\int_S \mathbf{Q}(\bar{x})d\bar{x} \qquad (1.11)$$

Here S is a region in the x_1, x_2 plane, which, in the limit, occupies the whole plane. We substitute the expression for $\mathbf{Q}(x)$ from (1.1) here and, taking into account the fact that, in view of the linearity of the problem, the function $F(x)$ can be represented in the form

$$F(\bar{x}) = \mathbf{A}(\bar{x})F^0, \quad \mathbf{A}(\bar{x}) = \begin{Vmatrix} A(\bar{x}) & \mu(\bar{x}) \\ \nu(\bar{x}) & \lambda(\bar{x}) \end{Vmatrix} \qquad (1.12)$$

we obtain

$$\langle \mathbf{Q}\rangle = n_0\mathbf{Q}F^0, \quad \mathbf{Q} = \langle \mathbf{Q}_S\rangle, \quad \mathbf{Q}_S = \int_S \mathbf{L}^1(\bar{x})\mathbf{A}(\bar{x})d\bar{x} \qquad (1.13)$$

Here n_0 is the numerical concentration of the fibres, the integral $\mathbf{Q}_s$ is taken over the cross-section S of each fibre, and the average $\langle \mathbf{Q}_s\rangle$ is calculated over the ensemble distribution of the operators $\mathbf{Q}_s$.

Henceforth we will assume that the mean deformation of the composite and the mean electric-field strength are fixed by the conditions of infinity, do not depend on the properties and concentration of the fibres and are identical with the constant external fields ε^0 and wE^0 ($\langle F\rangle = F^0$). In the same way as for the unconnected elastic problem [4], it can be shown that the action of the integral operators with kernels $\hat{\mathbf{P}}$ and $\mathbf{R}$ on the constants is given by the equations

$$\int \hat{P}(\bar{x}-\bar{x}')\langle \mathbf{Q}\rangle d\bar{x}' = 0, \quad \int \hat{\mathbf{R}}(\bar{x}-\bar{x}')\mathbf{M}^0\langle \mathbf{Q}\rangle d\bar{x}' = \langle \mathbf{Q}\rangle \qquad (1.14)$$

Hence, also from (1.10) and (1.13) we obtain the relations

$$\langle J\rangle = \mathbf{L}^*\langle F\rangle, \quad \mathbf{L}^* = \mathbf{L}^0 + n_0\mathbf{Q} \qquad (1.15)$$

where $\mathbf{L}^*$ is the electroelastic characteristics operator of the composite material.

Hence, the problem of averaging is equivalent to the problem of determining the effective electroelastic constants of the composite $\mathbf{L}^*$ and reduces to constructing the operator $\mathbf{Q}$, defined in (1.13). Here, to construct this operator, we use one of the self-consistent schemes (the effective-field method [4, 5]), based on solving the electroelastic problem for one isolated fibre in a homogeneous medium.

2. THE EFFECTIVE FIELD METHOD

In accordance with the main hypothesis of the effective-field method, we will assume that each fibre in the composite behaves as an isolated fibre in a homogeneous medium with the properties of the matrix $\mathbf{L}^0$, and the presence of the surrounding fibres is taken into account using effective external fields $F^* = [\varepsilon^*, E^*]$, which act on this fibre and which are not identical with $F^0 = [\varepsilon^0, E^0]$. The fields ε^* and E^* are assumed to be constant and the same for all the fibres.

Using this hypothesis, we can represent the function $\mathbf{Q}(x)$ in the form

$$\mathbf{Q}(x)=\mathbf{L}^1(\bar{x})\mathbf{A}^0(\bar{x})F^*\Omega(\bar{x}) \tag{2.1}$$

where $\Omega(\bar{x})$ is a characteristic function of the region in the x_1, x_2 plane occupied by the cross-section of the fibres, while the operator $\mathbf{A}^0(\bar{x})$ is found from the solution of the problem for a single fibre in a medium with the properties of the matrix $\mathbf{L}^0$ when acted upon by constant fields ε^* and $\bar{E}^*$.

In turn, the local external fields $F^* = [\varepsilon^*, E^*]$ at the point x, belonging to an arbitrary fibre ($\bar{x} \in \Omega$), are represented in the form

$$F^*(\bar{x})=F^0+\int \hat{\mathbf{P}}(\bar{x}-\bar{x}')\mathbf{L}^1(\bar{x}')\mathbf{A}^0(\bar{x}')F^*\Omega(\bar{x},\bar{x}')d\bar{x}' \tag{2.2}$$

The function $\Omega(\bar{x}, \bar{x})$ is defined by the relation

$$\Omega(\bar{x},\bar{x}')=\sum_{i\neq k}\Omega_i(\bar{x}'), \quad \bar{x}\in\Omega_k \tag{2.3}$$

where $\Omega_i(\bar{x})$ is a characteristic function of the region occupied by the cross-section of the ith fibre.

We will average relation (2.2) with the condition $\bar{x} \in \Omega$, assuming that the properties of the fibres are statistically independent of their position in space. As a result we obtain

$$\langle F^*(\bar{x})\,|\,\bar{x}\rangle=F^0+\left(n_0\int \hat{\mathbf{P}}(\bar{x}-\bar{x}')\mathbf{L}^A\Psi(\bar{x}-\bar{x}')d\bar{x}'\right)F^* \tag{2.4}$$

Here we have put

$$\mathbf{L}^A=\left\langle \int_S \mathbf{L}^1(\bar{x})\mathbf{A}^0(\bar{x})d\bar{x}\right\rangle, \quad \Psi(\bar{x}-\bar{x}')=\frac{1}{\langle\Omega(\bar{x})\rangle}\langle\Omega(\bar{x},\bar{x}')\,|\,\bar{x}\rangle \tag{2.5}$$

where $\langle\cdot|\bar{x}\rangle$ denotes averaging with the condition $x^- \in \Omega$. It follows from the definition of the function $\Omega(\bar{x}, \bar{x})$ that $\Psi(\bar{x})$ is a continuous function and $\Psi(0) = 0$, $\Psi(\bar{x}) \to 1$ when $|\bar{x}| \to \infty$. Henceforth we will assume that the cross-sections of the fibres are distributed isotropically in the x_1, x_2 plane. In this case $\Psi(\bar{x}) = \Psi(|\bar{x}|)$.

By identifying the average $\langle F^*(\bar{x})|\bar{x}\rangle$ with the effective field F^* for each inclusion, we obtain from (2.4)

$$F^*=\mathbf{D}F^0, \quad \mathbf{D}=(\mathbf{I}-n_0\mathbf{P}^0\mathbf{L}^A)^{-1}, \quad \mathbf{P}^0=\int\hat{\mathbf{P}}(\bar{x})[1-\Psi(\bar{x})]d\bar{x} \tag{2.6}$$

$$\mathbf{I}=\begin{Vmatrix} I_{ijkl} & 0 \\ 0 & \delta_{ik}\end{Vmatrix}, \quad I_{ijkl}=\delta_{i(k}\delta_{l)j}$$

Here the operator $\mathbf{P}^0$ is independent of the specific form of the function $\Psi(|x^-|)$, and in the case of an isotropic matrix with Lamé constants λ_0, μ_0 and permittivity ε_0, is defined by the expressions

$$\mathbf{P}^0=\begin{Vmatrix} P^0_{ijkl} & 0 \\ 0 & p^0_{ik}\end{Vmatrix} \tag{2.7}$$

$$P^0_{ijkl}=P^0_1\theta_{ij}\theta_{kl}+\frac{1}{2}P^0_2(\theta_{ik}\theta_{lj}+\theta_{il}\theta_{kj}-\theta_{ij}\theta_{kl})+$$

$$+\ \frac{1}{4}(P^0_2-P^0_1)(\theta_{ik}m_lm_j+\theta_{jk}m_lm_i+\theta_{il}m_km_j+\theta_{jl}m_km_i)$$

$$P^0_n=\frac{n-\varkappa_0}{4\mu_0}, \ n=1,2; \ \varkappa_0=\frac{\lambda_0+\mu_0}{\lambda_0+2\mu_0}, \ p^0_{ik}=\frac{1}{2\varepsilon_0}\theta_{ik}, \ \theta_{ij}=\delta_{ij}-m_im_j$$

where m_i are the components of the unit vector parallel to the fibres.

Using relations (2.6), we can write the following expression for the average of the function $\mathbf{Q}(x)$ (2.1)

$$\langle\mathbf{Q}\rangle=n_0\mathbf{Q}F^0, \quad \mathbf{Q}=\mathbf{L}^A\mathbf{D} \tag{2.8}$$

From this and from (1.14) we obtain the following expression for the effective electroelastic characteristics operator

$$\mathbf{L}^* = \mathbf{L}^0 + n_0\mathbf{L}^A\mathbf{D} \tag{2.9}$$

3. A COMPOSITE REINFORCED WITH SHUNTED FIBRES

We will assume that each fibre in the composite material consists of a core of radius a of piezoelectric ceramics, surrounded by a current-conducting shell (a shunt) with external radius b. We will assume the piezoelectric ceramics to be a transversally isotropic material with an axis of symmetry of the properties coinciding with the geometrical axis of the fibre. The external shell, like the matrix, is assumed to be isotropic.

Suppose the macroscopic loading of the composite reduces to a uniform deformation along the reinforcement direction $\langle\varepsilon_2\rangle$ and an electric field strength $\langle E_2\rangle$. The general method of solving the problem for a single isolated stratified non-uniform fibre, discussed for the elastic problem in [6], can also be used here for the connected electroelastic problem. However, in the case considered here of a fibre with a single coating and an axisymmetrical stress–strain state of the medium with the fibre, the solution of this problem is easier to obtain directly. Here the operator $\mathbf{A}^0(x^-)$ in (2.1) can be represented in the form

$$\mathbf{A}^0(\bar{x}) = \begin{Vmatrix} A(\bar{x}) & h(\bar{x}) \\ 0 & t^1 \end{Vmatrix} \tag{3.1}$$

Here

$$A_{ijkl}(\bar{x}) = \begin{cases} A^f_{ijkl}, \\ A^p_{ijkl}, \end{cases} \quad h(\bar{x}) = \begin{cases} h_f e_1^f \theta_{ij} m_k, & 0 \leq r \leq a \\ h_p e_1^f \theta_{ij} m_k, & a \leq r \leq b \end{cases}$$

$$A^s_{ijkl} = A_1^s \theta_{ij}\theta_{kl} + A_2^s \theta_{ij} m_k m_l + m_i m_j m_k m_l, \quad s = f, p$$

$$A_1^f = \frac{1}{2\Delta}(\lambda_0 + 2\mu_0)(\lambda_p + 2\mu_p), \quad A_1^p = \frac{1}{2\Delta}(\lambda_0 + 2\mu_0)(k_f + \mu_p)$$

$$A_2^f = -\frac{1}{2\Delta}[(l_f - \lambda_0)(\lambda_p + 2\mu_p) + \eta(l_f - \lambda_p)(\mu_0 - \mu_p)] \tag{3.2}$$

$$A_2^p = -\frac{1}{2\Delta}[(l_p - \lambda_0)(k_f + \mu_p) + (1 + \eta)(l_f - \lambda_p)(\mu_0 - \mu_p)]$$

$$h_f = \frac{1}{2\Delta}[\lambda_p + 2\mu_p + \eta(\mu_0 - \mu_p)], \quad h_p = -\frac{1-\eta}{2\Delta}(\mu_0 - \mu_p)$$

$$\eta = \frac{b^2 - a^2}{b^2}, \quad \Delta = (\lambda_p + 2\mu_p)(k_f + \mu_0) + \eta(k_f - \lambda_p - \mu_p)(\mu_0 - \mu_p)$$

In these expressions $t_{ij} = m_i m_j$, λ_0, μ_0, λ_p, μ_p are the Lamé constants of the materials of the matrix and the coating respectively, n_f, k_f and l_f are the moduli of elasticity of the transversely isotropic fibre, and e_1 and e_3 are its piezoelastic constants. These quantities can be expressed in the following way in terms of the "usual" double-index components of the tensors C^f_{ijkl} and E^{ef}_{ijkl}, most often encountered in the literature

$$n_f = C^f_{33}, \quad k_k = ½(C^f_{11} + C^f_{12}), \quad l_f = C^f_{13}, \quad e_1^f = e^f_{13}, \quad e_3^f = e^f_{33} \tag{3.3}$$

The composite as a whole is transversely isotropic. The above formulae enable us to obtain the following expressions for the three effective electroelastic characteristics of the material, which will later be necessary for analysing wave propagation

$$n^* = \lambda_0 + 2\mu_0 + p(n_A + \varkappa l_A^2) \tag{3.4}$$

$$e_3^* = p(e_3^A + \varkappa l_A e_1^A), \quad \varepsilon_3^* = \varepsilon_0 + p(e_3^A - \varkappa(e_1^A)^2)$$

Here

$$
\begin{aligned}
&\varkappa = 4pP_1^0(1-4pP_1^0k_A)^{-1}\\
&k_A = 2\eta(k_p-\lambda_0-\mu_0)A_1^p + 2(1-\eta)(k_f-\lambda_0-\mu_0)A_1^f\\
&l_A = 2\eta(l_p-\lambda_0)A_1^p + 2(1-\eta)(l_f-\lambda_0)A_1^f\\
&n_A = \eta[\lambda_p+2\mu_p-\lambda_0-2\mu_0+2(\lambda_p-\lambda_0)A_2^p]+\\
&+\ (1-\eta)[n_f-\lambda_0-2\mu_0+(l_f-\lambda_0)A_2^f]\\
&e_1^A = 2(1-\eta)e_1^fA_1^f,\quad e_3^A=(1-\eta)(e_3^f+2e_1^fA_2^f)\\
&\varepsilon_3^A = \eta(\varepsilon_p-\varepsilon_0)+(1-\eta)(e_3^f-\varepsilon_0)+2(1-\eta)(e_1^f)^2h_f
\end{aligned}
\tag{3.5}
$$

p is the volume density of the fibres, and ε_0, ε_p and ε_3 are the permittivities of the matrix, the coating and the fibre in the direction of its axis of symmetry. Note that the simple linear model, based on a "mixture rule" [2], leads to the following effective electroelastic characteristics

$$
\begin{aligned}
&n_{(l)}^* = \lambda_0+2\mu_0+p[\eta(\lambda_p+2\mu_p)-\lambda_0-2\mu_0+\\
&+\ (1-\eta)(n_f-\lambda_0-2\mu_0)],\quad e_{3(l)}^* = pe_3^f\\
&e_{3(l)}^* = \varepsilon_0+p[\eta(\varepsilon_p-\varepsilon_0)+(1-\eta)(e_3^f-\varepsilon_0)]
\end{aligned}
\tag{3.6}
$$

The difference between these quantities and the corresponding effective constants, presented in (3.4), is proportional to the square of the difference between the electroelastic constants of the components. This difference is unimportant if the characteristics of the components differ only slightly, but if there is a large contrast in their properties the differences between the quantities (3.4) and (3.6) may be considerable.

4. THE PROPAGATION OF A LONGITUDINAL ELASTIC WAVE IN A MEDIUM REINFORCED WITH SHUNTED PIEZOELECTRIC FIBRES

The equations of steady oscillations of such a composite, supplemented by the corollary of Ampère's law in the long-wave approximation, can be written in the form

$$
\begin{aligned}
&\operatorname{div}\langle\sigma(x)\rangle+\omega^2\rho^*\langle u(x)\rangle = 0\\
&\operatorname{div}(i\omega\langle D(x)\rangle-\beta^*\operatorname{grad}\langle\varphi(x)\rangle) = 0
\end{aligned}
\tag{4.1}
$$

Here $\langle u_i(x)\rangle$ is the amplitude value of the mean elastic displacement vector, $\langle\varphi(x)\rangle$ is the average potential of the electric field, ω is the frequency, β_{ij}^* is the tensor of the effective electrical conductivities of the medium, and ρ^* is its effective density

$$\rho^* = (1-\eta)\rho_0+p[\eta\rho_p+(1-\eta)\rho_f]$$

where ρ_0, ρ_p and ρ_f is the density of the matrix, the coating and the piezoelectric ceramics, respectively. Assuming that the electrical conductivity of the matrix and of the piezoelectric ceramic fibre are negligibly small compared with the electrical conductivity of the coating $\beta_{ij}^p = \beta_p\delta_{ij}$, we have

$$\beta_{ij}^* = p\eta\beta_p\delta_{ij}$$

By using the governing relations for the composite

$$\langle J(x)\rangle = \mathbf{L}^*\langle F(x)\rangle,\quad \mathbf{L}^* = \begin{Vmatrix} C^* & -e^* \\ e^{*T} & \varepsilon^* \end{Vmatrix}$$

we can write

$$\partial_j(C^*_{ijkl}\partial_l\langle u_k(x)\rangle + e^*_{ijk}\partial_k\langle\varphi(x)\rangle) + \rho^*\omega^2\langle u_i(x)\rangle = 0$$

$$\partial_j(i\omega e^{T*}_{jkl}\partial_l\langle u_k(x)\rangle - i\omega\varepsilon^*_{jk}\partial_k\langle\varphi(x)\rangle - \beta^*_{jk}\partial_k\langle\varphi(x)\rangle) = 0 \qquad (4.2)$$

We will seek a solution of this system in the form of plane waves, propagating along the reinforcement direction

$$\langle u_i(x)\rangle = Um_i e^{iq\mathbf{m}\cdot\mathbf{x}}, \quad \langle\varphi(x)\rangle = \Phi e^{iq\mathbf{m}\cdot\mathbf{x}}$$

where q is the wave number. Substituting these expressions into (4.2) we obtain a system of algebraic equations in U and Φ, whence follows a dispersion relation, which can be written in the form [2]

$$q^2 n^*(\omega) = \rho^*\omega^2, \quad n^*(\omega) = n'(\omega) + in''(\omega)$$

$$n'(\omega) = n^*\left[1 + \frac{\delta(\omega\tau)^2}{1+(\omega\tau)^2}\right], \quad n''(\omega) = \frac{n^*\delta\omega\tau}{1+(\omega\tau)^2}$$

$$\delta = \frac{e_3^{*2}}{n^*\varepsilon_3^*}, \quad \tau = \frac{\varepsilon_3^*}{p\eta\beta_p}$$

Hence, the wave number q is complex

$$q(\omega) = q_R(\omega) + iq_I(\omega)$$

$$q_R = \frac{\omega}{\upsilon^*(\omega)}, \quad q_I = \frac{\omega}{\upsilon^*(\omega)}\operatorname{tg}\frac{\psi}{2}; \quad \upsilon^*(\omega) = \sqrt{\frac{|n^*(\omega)|}{\rho^*}}\sec\frac{\psi}{2}$$

Its real part defines the dispersion of the phase velocity $\upsilon^*(\omega)$ while the imaginary part defines the frequency dependence of the attenuation factor, referred to unit length. In these expressions ψ is the phase delay angle, which is defined in terms of the loss tangent

$$\operatorname{tg}\psi = \frac{\delta\omega\tau}{1+(1+\delta)(\omega\tau)^2}$$

Hence we can obtain the maximum loss tangent

$$(\operatorname{tg}\psi)_{\max} = \frac{\delta}{2\sqrt{1+\delta}}$$

Example. We will consider a numerical example for the following values of the electroelastic characteristics of the components [3] (the fibre is made of PZT-5A ceramics of diameter 30 μm with a copper coating 1 μm thick in a polymer matrix)

$k_f = 98.2$; $l_f = 75.2$; $n_f = 111$ GPa
$e_1^f = -5.4$; $e_3^f = 15.8$; $\varepsilon_3^f = 7.34$; $p = 0.3$
$\lambda_p = 120$; $\mu_p = 40$ GPa, $\varepsilon_p = 0.9 \times 10^{-11}$, $\beta_p = 6.2 \times 10^{-5}\,(\Omega m)^{-1}$, $\eta = 0.06$
$\lambda_0 = 4.4$, $\mu_0 = 1.8$ GPa, $\varepsilon_0 = 3.7 \times 10^{-11}$

The maximum loss tangent calculated for these values of the constants is 0.183 at a frequency of 66 Hz. The value calculated using the simplified formulae (3.6) proposed in [2] is equal to 0.13.

REFERENCES

1. HAGOOD, N. W. and FLOTOW, A. Vn., Damping of structural vibrations with piezoelectric materials and passive electrical networks. *J. Sound Vibrat.*, 1991, **146**, 2, 243–268.
2. MAYER, R., MOSES, P., SHROUT, T., WITHAM, J., LESIEUTRE, G. A. and YARLAGADDA, S., Passive vibration damping materials; Piezoelectric ceramic composites for vibration damping application. Annual Technical Report, Grant No. N00014-92-J-1391. February 1994, 15.
3. LESIEUTRE, G. A., YARLAGADDA, S., YOSHIKAWA, S., KURTZ, S. K. and XY, Q. C., Passively damped structural composite materials using resistively shunted piezoceramic fibres. *J. Mater. Enging and Performance*, 1993, **2**, 6, 27–32.

4. KANAUN, S. K. and LEVIN, V. M., *The Effective Field Method in Mechanics of Matrix Composite Materials. Advances in Mathematical Modeling of Composite Materials.* World Scientific Publ. Comp., Singapore, 1994, 1–58.
5. LEVIN, V. M., The overall properties of piezoactive matrix composite materials. *Continuum Models and Discrete Systems: Proc. CMDS8*. World Scientific Publ. Comp., Singapore, 1996, 225–232.
6. KANAUN, S. K. and KUDRYAVTSEVA, L. T., The elastic and thermoelastic characteristics of composites reinforced with unidirectional layered fibres. *Prikl. Mat. Mekh.*, 1988, **52**, 5, 798–807.

Translated by R.C.G.